AP Calculus AB Summer Assignment 2022

Name: ______________________

Date: _____________

About the class:

The AP Calculus AB course is the equivalent of a 1 semester college level course in Calculus. Like other college classes, students are required to work independently outside of class time. The course is designed to prepare students for the AP Exam in May, and a rigorous pace is necessary to do so. Students should expect to work hard and learn a lot. Students will be taking notes and attempting a few exercises at home so that class time can be dedicated to breaking down the difficult concepts and real problems.

About the Summer Assignment:

For this summer assignment, you will do some reading and some practice exercises. Most (if not all) of the exercises should be review for you. If you do not remember how to do the exercises, you should research how to do them on your own, but you can contact me as well. I will send out the answer key close to the end of the summer. ***It is imperative that you check your answers, and that you show your work. (No work = no credit)***

It is due the first day of classes and will be a significant portion of your first quarter grade. You will be tested on the material within the first 2 weeks of classes.

Other notes:

- You can email me with questions over the summer if you wish. If certain concepts and exercises trouble you, that's ok, but you are expected to at least try.
- Do not rely on the calculator to do work which you could do in your head or on paper. A general rule to follow is to always try to solve problems without the calculator first.
- Make sure to memorize the unit circle values and not rely on notes or the circle chart.
    - Ask any former calculus student and they will tell you that one of the hardest parts of calculus is the underlying algebra and trigonometry.
- Try to pace yourself over the summer for completion of this assignment. Making a strong effort to complete this with accuracy will prepare you for the rigor expected in AP Calc.

AP Calculus AB Summer Assignment 2022

Name: ______________________

Date: _____________

Actual Assignment:

1.) Attached packet from FlippedMath.com

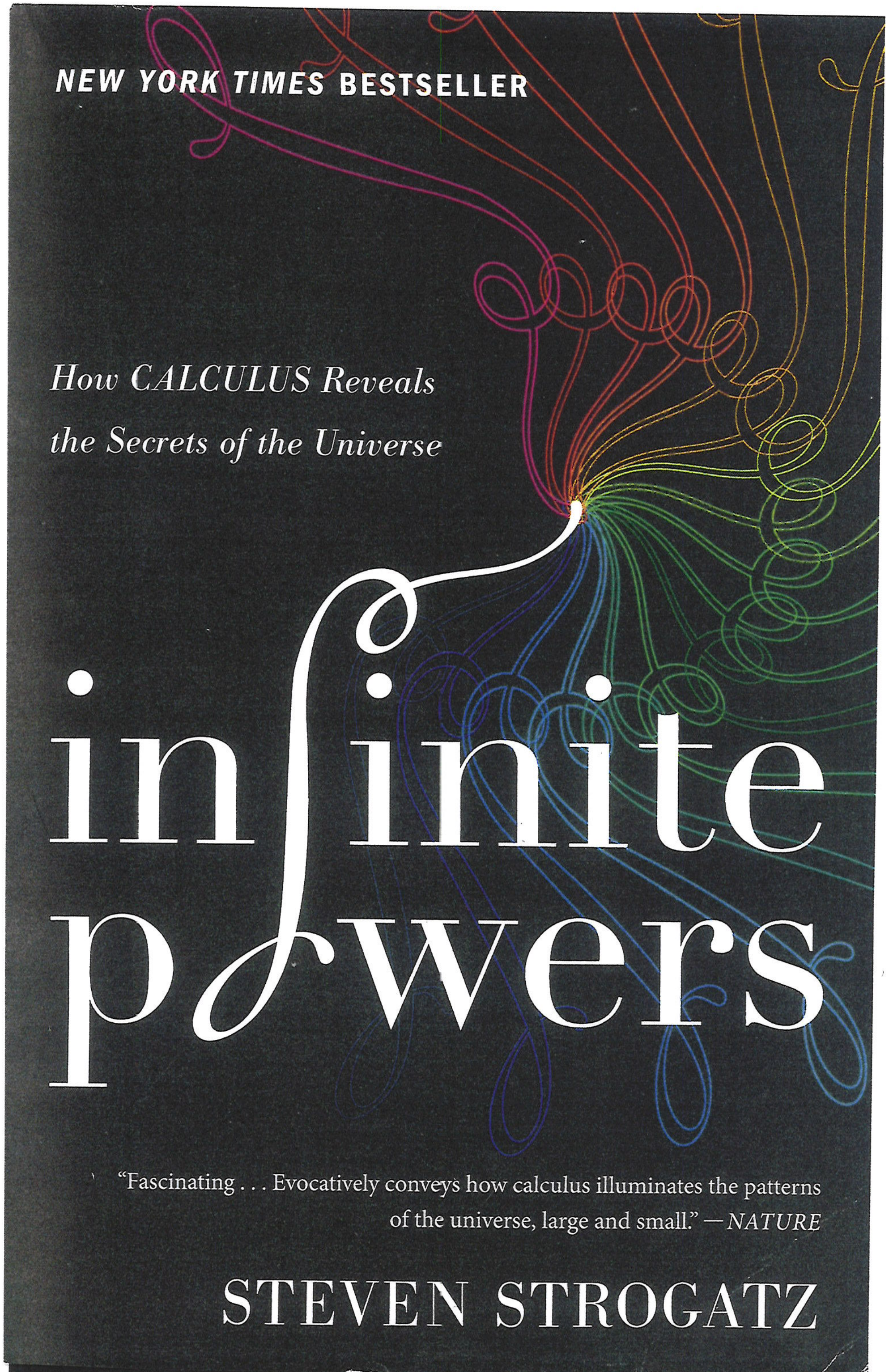
2.) Attached worksheet on Factoring - ALL problems.

3.) Questions on the reading (see attached pages from the Introduction and Chapter 1 of *Infinite Powers* by Steven Strogatz):

   a.) "Thus, calculus proceeds in two phases: cutting and rebuilding. In mathematical terms, the cutting process always involves infinitely fine subtraction, which is used to quantify the differences between the parts. Accordingly, this half of the subject is called *differential* calculus. The reassembly process always involves infinite addition, which integrates the parts back into the original whole. This half of the subject is called *integral* calculus."
   Describe the context and significance of this quote. What does it tell you about what you will be learning in the year to come? Which half do you anticipate being more difficult?

   b.) "So infinity times zero would have to be both the circumference and double the circumference. What nonsense! There simply is no consistent way to define infinity times zero, and so there is no sensible way to regard a circle as an infinite polygon."
   Describe the context of this quote and explain the difference between completed and potential infinity. What is so alluring about infinity and what is so dangerous about it?

   c.) What section or quote was most profound/interesting/significant to you and why?



Introduction

Without calculus, we wouldn't have cell phones, computers, or microwave ovens. We wouldn't have radio. Or television. Or ultrasound for expectant mothers, or GPS for lost travelers. We wouldn't have split the atom, unraveled the human genome, or put astronauts on the moon. We might not even have the Declaration of Independence.

It's a curiosity of history that the world was changed forever by an arcane branch of mathematics. How could it be that a theory originally about shapes ultimately reshaped civilization?

The essence of the answer lies in a quip that the physicist Richard Feynman made to the novelist Herman Wouk when they were discussing the Manhattan Project. Wouk was doing research for a big novel he hoped to write about World War II, and he went to Caltech to interview physicists who had worked on the bomb, one of whom was Feynman. After the interview, as they were parting, Feynman asked Wouk if he knew calculus. No, Wouk admitted, he didn't. "You had better learn it," said Feynman. "It's the language God talks."

For reasons nobody understands, the universe is deeply mathematical. Maybe God made it that way. Or maybe it's the only way a universe with us in it could be, because nonmathematical universes can't harbor life intelligent enough to ask the question. In any case, it's a mysterious and marvelous fact that our universe obeys laws of nature that always turn out to be expressible in the language of calculus as sentences called differential equations. Such equations describe the difference between something right now and the same thing an instant later or between something right here and the same thing infinitesimally close by. The details differ depending on what part of nature we're talking about, but the structure of the laws is always the same. To put this awesome assertion another way, there seems to be something like a code to the universe, an operating system that animates everything from moment to moment and place to place. Calculus taps into this order and expresses it.

Isaac Newton was the first to glimpse this secret of the universe. He found that the orbits of the planets, the rhythm of the tides, and the trajectories of cannonballs could all be described, explained, and predicted by a small set of differential equations. Today we call them Newton's laws of motion and gravity. Ever since Newton, we have found that the same pattern holds whenever we uncover a new part of the universe. From the old elements of earth, air, fire, and water to the latest in electrons, quarks, black holes, and superstrings, every inanimate thing in the universe bends to the rule of differential equations. I bet this is what Feynman meant when he said that calculus is the language God talks. If anything deserves to be called the secret of the universe, calculus is it.

By inadvertently discovering this strange language, first in a corner of geometry and later in the code of the universe, then by learning to speak it fluently and decipher its idioms and nuances, and finally by harnessing its forecasting powers, humans have used calculus to remake the world.

That's the central argument of this book.

If it's right, it means the answer to the ultimate question of life, the universe, and everything is not 42, with apologies to fans of Douglas Adams and *The Hitchhiker's Guide to the Galaxy*. But Deep Thought was on the right track: the secret of the universe is indeed mathematical.

Calculus for Everyone

Feynman's quip about God's language raises many profound questions. What is calculus? How did humans figure out that God speaks

it (or, if you prefer, that the universe runs on it)? What are differential equations and what have they done for the world, not just in Newton's time but in our own? Finally, how can any of these stories and ideas be conveyed enjoyably and intelligibly to readers of goodwill like Herman Wouk, a very thoughtful, curious, knowledgeable person with little background in advanced math?

In a coda to the story of his encounter with Feynman, Wouk wrote that he didn't get around to even trying to learn calculus for fourteen years. His big novel ballooned into two big novels —*Winds of War* and *War and Remembrance*, each about a thousand pages. Once those were finally done, he tried to teach himself by reading books with titles like *Calculus Made Easy*—but no luck there. He poked around in a few textbooks, hoping, as he put it, "to come across one that might help a mathematical ignoramus like me, who had spent his college years in the humanities—i.e., literature and philosophy—in an adolescent quest for the meaning of existence, little knowing that calculus, which I had heard of as a difficult bore leading nowhere, was the language God talks." After the textbooks proved impenetrable, he hired an Israeli math tutor, hoping to pick up a little calculus and improve his spoken Hebrew on the side, but both hopes ran aground. Finally, in desperation, he audited a high-school calculus class, but he fell too far behind and had to give up after a couple of months. The kids clapped for him on his way out. He said it was like sympathy applause for a pitiful showbiz act.

I've written *Infinite Powers* in an attempt to make the greatest ideas and stories of calculus accessible to everyone. It shouldn't be necessary to endure what Herman Wouk did to learn about this landmark in human history. Calculus is one of humankind's most inspiring collective achievements. It isn't necessary to learn how to do calculus to appreciate it, just as it isn't necessary to learn how to prepare fine cuisine to enjoy eating it. I'm going to try to explain everything we'll need with the help of pictures, metaphors, and anecdotes. I'll also walk us through some of the finest equations and proofs ever created, because how could we visit a gallery without seeing its masterpieces? As for Herman Wouk, he is 103 years old as of this writing. I don't know if he's learned calculus yet, but if not, this one's for you, Mr. Wouk.

The World According to Calculus

As should be obvious by now, I'll be giving an applied mathematician's take on the story and significance of calculus. A historian of mathematics would tell it differently. So would a pure mathematician. What fascinates me as an applied mathematician is the push and pull between the real world around us and the ideal world in our heads. Phenomena out there guide the mathematical questions we ask; conversely, the math we imagine sometimes foreshadows what actually happens out there in reality. When it does, the effect is uncanny.

To be an applied mathematician is to be outward-looking and intellectually promiscuous. To those in my field, math is not a pristine, hermetically sealed world of theorems and proofs echoing back on themselves. We embrace all kinds of subjects: philosophy, politics, science, history, medicine, all of it. That's the story I want to tell—the world according to calculus.

This is a much broader view of calculus than usual. It encompasses the many cousins and spinoffs of calculus, both within mathematics and in the adjacent disciplines. Since this big-tent view is unconventional, I want to make sure it doesn't cause any confusion. For example, when I said earlier that without calculus we wouldn't have computers and cell phones and so on, I certainly didn't mean to suggest that calculus produced all these wonders by itself. Far from it. Science and technology were essential partners—and arguably the stars of the show. My point is merely that calculus has also played a crucial role, albeit often a supporting one, in giving us the world we know today.

Take the story of wireless communication. It began with the discovery of the laws of electricity and magnetism by scientists like Michael Faraday and André-Marie Ampère. Without their observations and tinkering, the crucial facts about magnets, electrical currents, and their invisible force fields would have remained unknown, and the possibility of wireless communication would never have been realized. So, obviously, experimental physics was indispensable here.

But so was calculus. In the 1860s, a Scottish mathematical physicist named James Clerk Maxwell recast the experimental laws of electricity and magnetism into a symbolic form that could be fed into the maw of calculus. After some churning, the maw disgorged an equation that didn't make sense. Apparently something was missing in the physics. Maxwell suspected that Ampère's law was the culprit. He tried patching it up by including a new term in his equation—a hypothetical current that would resolve the contradiction—and then let calculus churn again. This time it spat out a sensible result, a simple, elegant wave equation much like the equation that describes the spread of ripples on a pond. Except Maxwell's result was predicting a new kind of wave, with electric and magnetic fields dancing together in a pas de deux. A changing electric field would generate a changing magnetic field, which in turn would regenerate the electric field, and so on, each field bootstrapping the other forward, propagating together as a wave of traveling energy. And when Maxwell calculated the speed of this wave, he found—in what must have been one of the greatest Aha! moments in history—that it moved at the speed of light. So he used calculus not only to predict the existence of electromagnetic waves but also to solve an age-old mystery: What was the nature of light? Light, he realized, was an electromagnetic wave.

Maxwell's prediction of electromagnetic waves prompted an experiment by Heinrich Hertz in 1887 that proved their existence. A decade later, Nikola Tesla built the first radio communication system, and five years after that, Guglielmo Marconi transmitted the first wireless messages across the Atlantic. Soon came television, cell phones, and all the rest.

Clearly, calculus could not have done this alone. But equally clearly, none of it would have happened *without* calculus. Or, perhaps more accurately, it might have happened, but only much later, if at all.

Calculus Is More than a Language

The story of Maxwell illustrates a theme we'll be seeing again and again. It's often said that mathematics is the language of science. There's a great deal of truth to that. In the case of electromagnetic waves, it was a key first step for Maxwell to translate the laws that had been discovered experimentally into equations phrased in the language of calculus.

But the language analogy is incomplete. Calculus, like other forms of mathematics, is much more than a language; it's also an incredibly powerful system of reasoning. It lets us transform one equation into another by performing various symbolic operations on them, operations subject to certain rules. Those rules are deeply rooted in logic, so even though it might seem like we're just shuffling symbols around, we're actually constructing long chains of logical inference. The symbol shuffling is useful shorthand, a convenient way to build arguments too intricate to hold in our heads.

If we're lucky and skillful enough—if we transform the equations in just the right way—we can get them to reveal their hidden implications. To a mathematician, the process feels almost palpable. It's as if we're manipulating the equations, massaging them, trying to relax them enough so that they'll spill their secrets. We want them to open up and talk to us.

Creativity is required, because it often isn't clear which manipulations to perform. In Maxwell's case, there were countless ways to transform his equations, all of which would have been logically acceptable but only some of which would have been scientifically revealing. Given that he didn't even know what he was searching for, he might easily have gotten nothing out of his equations but incoherent mumblings (or the symbolic equivalent thereof). Fortunately, however, they did have a secret to reveal. With just the right prodding, they gave up the wave equation.

At that point the linguistic function of calculus took over again. When Maxwell translated his abstract symbols back into reality, they predicted that electricity and magnetism could propagate together

as a wave of invisible energy moving at the speed of light. In a matter of decades, this revelation would change the world.

Unreasonably Effective

It's eerie that calculus can mimic nature so well, given how different the two domains are. Calculus is an imaginary realm of symbols and logic; nature is an actual realm of forces and phenomena. Yet somehow, if the translation from reality into symbols is done artfully enough, the logic of calculus can use one real-world truth to generate another. Truth in, truth out. Start with something that is empirically true and symbolically formulated (as Maxwell did with the laws of electricity and magnetism), apply the right logical manipulations, and out comes another empirical truth, possibly a new one, a fact about the universe that nobody knew before (like the existence of electromagnetic waves). In this way, calculus lets us peer into the future and predict the unknown. That's what makes it such a powerful tool for science and technology.

But why should the universe respect the workings of any kind of logic, let alone the kind of logic that we puny humans can muster? This is what Einstein marveled at when he wrote, "The eternal mystery of the world is its comprehensibility." And it's what Eugene Wigner meant in his essay "On the Unreasonable Effectiveness of Mathematics in the Natural Sciences" when he wrote, "The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve."

This sense of awe goes way back in the history of mathematics. According to legend, Pythagoras felt it around 550 BCE when he and his disciples discovered that music was governed by the ratios of whole numbers. For instance, imagine plucking a guitar string. As the string vibrates, it emits a certain note. Now put your finger on a fret exactly halfway up the string and pluck it again. The vibrating part of the string is now half as long as it used to be—a ratio of 1 to 2—and it sounds precisely an octave higher than the original note (the musical distance from one *do* to the next in the *do-re-mi-fa-sol-la-ti-do* scale). If instead the vibrating string is ⅔ of its original length, the note it makes goes up by a fifth (the interval from *do* to *sol;* think of the first two notes of the Star Wars theme). And if the vibrating part is ¾ as long as it was before, the note goes up by a fourth (the interval between the first two notes of "Here Comes the Bride"). The ancient Greek musicians knew about the melodic concepts of octaves, fourths, and fifths and considered them beautiful. This unexpected link between music (the harmony of this world) and numbers (the harmony of an imagined world) led the Pythagoreans to the mystical belief that *all* is number. They are said to have believed that even the planets in their orbits made music, the music of the spheres.

Ever since then, many of history's greatest mathematicians and scientists have come down with cases of Pythagorean fever. The astronomer Johannes Kepler had it bad. So did the physicist Paul Dirac. As we'll see, it drove them to seek, and to dream, and to long for the harmonies of the universe. In the end it pushed them to make their own discoveries that changed the world.

The Infinity Principle

To help you understand where we're headed, let me say a few words about what calculus is, what it wants (metaphorically speaking), and what distinguishes it from the rest of mathematics. Fortunately, a single big, beautiful idea runs through the subject from beginning to end. Once we become aware of this idea, the structure of calculus falls into place as variations on a unifying theme.

Alas, most calculus courses bury the theme under an avalanche of formulas, procedures, and computational tricks. Come to think of it, I've never seen it spelled out anywhere even though it's part of calculus culture and every expert knows it implicitly. Let's call it the Infinity Principle. It will guide us on our journey just as it guided the development of calculus itself, conceptually as well as historically. I'm tempted to state it right now, but at this point it would sound like mumbo jumbo. It will be easier to appreciate if we inch

our way up to it by asking what calculus wants . . . and how it gets what it wants.

In a nutshell, calculus wants to make hard problems simpler. It is utterly obsessed with simplicity. That might come as a surprise to you, given that calculus has a reputation for being complicated. And there's no denying that some of its leading textbooks exceed a thousand pages and weigh as much as bricks. But let's not be judgmental. Calculus can't help how it looks. Its bulkiness is unavoidable. It looks complicated because it's trying to tackle complicated problems. In fact, it has tackled and solved some of the most difficult and important problems our species has ever faced.

Calculus succeeds by breaking complicated problems down into simpler parts. That strategy, of course, is not unique to calculus. All good problem-solvers know that hard problems become easier when they're split into chunks. The truly radical and distinctive move of calculus is that it takes this divide-and-conquer strategy to its utmost extreme—*all the way out to infinity*. Instead of cutting a big problem into a handful of bite-size pieces, it keeps cutting and cutting relentlessly until the problem has been chopped and pulverized into its tiniest conceivable parts, leaving infinitely many of them. Once that's done, it solves the original problem for all the tiny parts, which is usually a much easier task than solving the initial giant problem. The remaining challenge at that point is to put all the tiny answers back together again. That tends to be a much harder step, but at least it's not as difficult as the original problem was.

Thus, calculus proceeds in two phases: cutting and rebuilding. In mathematical terms, the cutting process always involves infinitely fine subtraction, which is used to quantify the differences between the parts. Accordingly, this half of the subject is called *differential* calculus. The reassembly process always involves infinite addition, which integrates the parts back into the original whole. This half of the subject is called *integral* calculus.

This strategy can be used on anything that we can imagine slicing endlessly. Such infinitely divisible things are called *continua* and are said to be *continuous*, from the Latin roots *con* (together with) and *tenere* (hold), meaning uninterrupted or holding together. Think of the rim of a perfect circle, a steel girder in a suspension bridge, a bowl of soup cooling off on the kitchen table, the parabolic trajectory of a javelin in flight, or the length of time you have been alive. A shape, an object, a liquid, a motion, a time interval—all of them are grist for the calculus mill. They're all continuous, or nearly so.

Notice the act of creative fantasy here. Soup and steel are not really continuous. At the scale of everyday life, they appear to be, but at the scale of atoms or superstrings, they're not. Calculus ignores the inconvenience posed by atoms and other uncuttable entities, not because they don't exist but because it's useful to pretend that they don't. As we'll see, calculus has a penchant for useful fictions.

More generally, the kinds of entities modeled as continua by calculus include almost anything one can think of. Calculus has been used to describe how a ball rolls continuously down a ramp, how a sunbeam travels continuously through water, how the continuous flow of air around a wing keeps a hummingbird or an airplane aloft, and how the concentration of HIV virus particles in a patient's bloodstream plummets continuously in the days after he or she starts combination-drug therapy. In every case the strategy remains the same: split a complicated but continuous problem into infinitely many simpler pieces, then solve them separately and put them back together.

Now we're finally ready to state the big idea.

The Infinity Principle

To shed light on any continuous shape, object, motion, process, or phenomenon—no matter how wild and complicated it may appear—reimagine it as an infinite series of simpler parts, analyze those, and then add the results back together to make sense of the original whole.

The Golem of Infinity

The rub in all of this is the need to cope with infinity. That's easier said than done. Although the carefully controlled use of infinity

is the secret to calculus and the source of its enormous predictive power, it is also calculus's biggest headache. Like Frankenstein's monster or the golem in Jewish folklore, infinity tends to slip out of its master's control. As in any tale of hubris, the monster inevitably turns on its maker.

The creators of calculus were aware of the danger but still found infinity irresistible. Sure, occasionally it ran amok, leaving paradox, confusion, and philosophical havoc in its wake. Yet after each of these episodes, mathematicians always managed to subdue the monster, rationalize its behavior, and put it back to work. In the end, everything always turned out fine. Calculus gave the right answers, even when its creators couldn't explain why. The desire to harness infinity and exploit its power is a narrative thread that runs through the whole twenty-five-hundred-year story of calculus.

All this talk of desire and confusion might seem out of place, given that mathematics is usually portrayed as exact and impeccably rational. It is rational, but not always initially. Creation is intuitive; reason comes later. In the story of calculus, more than in other parts of mathematics, logic has always lagged behind intuition. This makes the subject feel especially human and approachable, and its geniuses more like the rest of us.

Curves, Motion, and Change

The Infinity Principle organizes the story of calculus around a methodological theme. But calculus is as much about mysteries as it is about methodology. Three mysteries above all have spurred its development: the mystery of curves, the mystery of motion, and the mystery of change.

The fruitfulness of these mysteries has been a testament to the value of pure curiosity. Puzzles about curves, motion, and change might seem unimportant at first glance, maybe even hopelessly esoteric. But because they touch on such rich conceptual issues and because mathematics is so deeply woven into the fabric of the universe, the solution to these mysteries has had far-reaching impacts on the course of civilization and on our everyday lives. As we'll see in the chapters ahead, we reap the benefits of these investigations whenever we listen to music on our phones, breeze through the line at the supermarket thanks to a laser checkout scanner, or find our way home with a GPS gadget.

It all started with the mystery of curves. Here I'm using the term *curves* in a very loose sense to mean any sort of curved line, curved surface, or curved solid—think of a rubber band, a wedding ring, a floating bubble, the contours of a vase, or a solid tube of salami. To keep things as simple as possible, the early geometers typically concentrated on abstract, idealized versions of curved shapes and ignored thickness, roughness, and texture. The surface of a mathematical sphere, for instance, was imagined to be an infinitesimally thin, smooth, perfectly round membrane with none of the thickness, bumpiness, or hairiness of a coconut shell. Even under these idealized assumptions, curved shapes posed baffling conceptual difficulties because they weren't made of straight pieces. Triangles and squares were easy. So were cubes. They were composed of straight lines and flat pieces of planes joined together at a small number of corners. It wasn't hard to figure out their perimeters or surface areas or volumes. Geometers all over the world—in ancient Babylon and Egypt, China and India, Greece and Japan—knew how to solve problems like these. But round things were brutal. No one could figure out how much surface area a sphere had or how much volume it could hold. Even finding the circumference and area of a circle was an insurmountable problem in the old days. There was no way to get started. There were no straight pieces to latch onto. Anything that was curved was inscrutable.

So this is how calculus began. It grew out of geometers' curiosity and frustration with roundness. Circles and spheres and other curved shapes were the Himalayas of their era. It wasn't that they posed important practical issues, at least not at first. It was simply a matter of the human spirit's thirst for adventure. Like explorers climbing Mount Everest, geometers wanted to solve curves because they were there.

The breakthrough came from insisting that curves *were* actually made of straight pieces. It wasn't true, but one could pretend that it was. The only hitch was that those pieces would then have to be infinitesimally small and infinitely numerous. Through this fantastic conception, integral calculus was born. This was the earliest use of the Infinity Principle. The story of how it developed will occupy us for several chapters, but its essence is already there, in embryonic form, in a simple, intuitive insight: If we zoom in closely enough on a circle (or anything else that is curved and smooth), the portion of it under the microscope begins to look straight and flat. So in principle, at least, it should be possible to calculate whatever we want about a curved shape by adding up all the straight little pieces. Figuring out exactly how to do this—no easy feat—took the efforts of the world's greatest mathematicians over many centuries. Collectively, however, and sometimes through bitter rivalries, they eventually began to make headway on the riddle of curves. Spinoffs today, as we'll see in chapter 2, include the math needed to draw realistic-looking hair, clothing, and faces of characters in computer-animated movies and the calculations required for doctors to perform facial surgery on a virtual patient before they operate on the real one.

The quest to solve the mystery of curves reached a fever pitch when it became clear that curves were much more than geometric diversions. They were a key to unlocking the secrets of nature. They arose naturally in the parabolic arc of a ball in flight, in the elliptical orbit of Mars as it moved around the sun, and in the convex shape of a lens that could bend and focus light where it was needed, as was required for the burgeoning development of microscopes and telescopes in late Renaissance Europe.

And so began the second great obsession: a fascination with the mysteries of motion on Earth and in the solar system. Through observation and ingenious experiments, scientists discovered tantalizing numerical patterns in the simplest moving things. They measured the swinging of a pendulum, clocked the accelerating descent of a ball rolling down a ramp, and charted the stately procession of planets across the sky. The patterns they found enraptured them—indeed, Johannes Kepler fell into a state of self-described "sacred frenzy" when he found his laws of planetary motion—because those patterns seemed to be signs of God's handiwork. From a more secular perspective, the patterns reinforced the claim that nature was deeply mathematical, just as the Pythagoreans had maintained. The only catch was that nobody could explain the marvelous new patterns, at least not with the existing forms of math. Arithmetic and geometry were not up to the task, even in the hands of the greatest mathematicians.

The trouble was that the motions weren't steady. A ball rolling down a ramp kept changing its speed, and a planet revolving around the sun kept changing its direction of travel. Worse yet, the planets moved faster when they got close to the sun and slowed down as they receded from it. There was no known way to deal with motion that kept changing in ever-changing ways. Earlier mathematicians had worked out the mathematics of the most trivial kind of motion, namely, motion at a constant speed where distance equals rate times time. But when speed changed and kept on changing continuously, all bets were off. Motion was proving to be as much of a conceptual Mount Everest as curves were.

As we'll see in the middle chapters of this book, the next great advances in calculus grew out of the quest to solve the mystery of motion. The Infinity Principle came to the rescue, just as it had for curves. This time the act of wishful fantasy was to pretend that motion at a changing speed was made up of infinitely many, infinitesimally brief motions at a *constant* speed. To visualize what this would mean, imagine being in a car with a jerky driver at the wheel. As you anxiously watch the speedometer, it moves up and down with every jerk. But over a millisecond, even the jerkiest driver can't make the speedometer needle move by much. And over an interval much shorter than that—an infinitesimal time interval—the needle won't move at all. Nobody can tap the gas pedal that fast.

These ideas coalesced in the younger half of calculus, differential calculus. It was precisely what was needed to work with the infinitesimally small changes of time and distance that arose in the study of ever-changing motion as well as with the infinitesimal straight pieces of curves that arose in analytic geometry, the newfangled study of curves defined by algebraic equations that was all the rage in the first half of the 1600s. Yes, at one time, algebra was a craze, as we'll see.

Its popularity was a boon for all fields of mathematics, including geometry, but it also created an unruly jungle of new curves to explore. Thus, the mysteries of curves and motion collided. They were now both at the center stage of calculus in the mid-1600s, banging into each other, creating mathematical mayhem and confusion. Out of the tumult, differential calculus began to flower, but not without controversy. Some mathematicians were criticized for playing fast and loose with infinity. Others derided algebra as a scab of symbols. With all the bickering, progress was fitful and slow.

And then a child was born on Christmas Day. This young messiah of calculus was an unlikely hero. Born premature and fatherless and abandoned by his mother at age three, he was a lonesome boy with dark thoughts who grew into a secretive, suspicious young man. Yet Isaac Newton would make a mark on the world like no one before or since.

First, he solved the holy grail of calculus: he discovered how to put the pieces of a curve back together again—and how to do it easily, quickly, and systematically. By combining the symbols of algebra with the power of infinity, he found a way to represent any curve as a sum of infinitely many simpler curves described by powers of a variable x, like x^2, x^3, x^4, and so on. With these ingredients alone, he could cook up any curve he wanted by putting in a pinch of x and a dash of x^2 and a heaping tablespoon of x^3. It was like a master recipe and a universal spice rack, butcher shop, and vegetable garden, all rolled into one. With it he could solve any problem about shapes or motions that had ever been considered.

Then he cracked the code of the universe. Newton discovered that motion of any kind always unfolds one infinitesimal step at a time, steered from moment to moment by mathematical laws written in the language of calculus. With just a handful of differential equations (his laws of motion and gravity), he could explain everything from the arc of a cannonball to the orbits of the planets. His astonishing "system of the world" unified heaven and earth, launched the Enlightenment, and changed Western culture. Its impact on the philosophers and poets of Europe was immense. He even influenced Thomas Jefferson and the writing of the Declaration of

Independence, as we'll see. In our own time, Newton's ideas underpinned the space program by providing the mathematics necessary for trajectory design, the work done at NASA by African-American mathematician Katherine Johnson and her colleagues (the heroines of the book and hit movie *Hidden Figures*).

With the mysteries of curves and motion now settled, calculus moved on to its third lifelong obsession: the mystery of change. It's a cliché, but it's true all the same—nothing is constant but change. It's rainy one day and sunny the next. The stock market rises and falls. Emboldened by the Newtonian paradigm, the later practitioners of calculus asked: Are there laws of change similar to Newton's laws of motion? Are there laws for population growth, the spread of epidemics, and the flow of blood in an artery? Can calculus be used to describe how electrical signals propagate along nerves or to predict the flow of traffic on a highway?

By pursuing this ambitious agenda, always in cooperation with other parts of science and technology, calculus has helped make the world modern. Using observation and experiment, scientists worked out the laws of change and then used calculus to solve them and make predictions. For example, in 1917 Albert Einstein applied calculus to a simple model of atomic transitions to predict a remarkable effect called stimulated emission (which is what the *s* and *e* stand for in *laser*, an acronym for *light amplification by stimulated emission of radiation*). He theorized that under certain circumstances, light passing through matter could stimulate the production of more light at the same wavelength and moving in the same direction, creating a cascade of light through a kind of chain reaction that would result in an intense, coherent beam. A few decades later, the prediction proved to be accurate. The first working lasers were built in the early 1960s. Since then, they have been used in everything from compact-disc players and laser-guided weaponry to supermarket bar-code scanners and medical lasers.

The laws of change in medicine are not as well understood as those in physics. Yet even when applied to rudimentary models, calculus has been able to make lifesaving contributions. For example, in chapter 8 we'll see how a differential-equation model developed by

an immunologist and an AIDS researcher played a part in shaping the modern three-drug combination therapy for patients infected with HIV. The insights provided by the model overturned the prevailing view that the virus was lying dormant in the body; in fact, it was in a raging battle with the immune system every minute of every day. With the new understanding that calculus helped provide, HIV infection has been transformed from a near-certain death sentence to a manageable chronic disease—at least for those with access to combination-drug therapy.

Admittedly, some aspects of our ever-changing world lie beyond the approximations and wishful thinking inherent in the Infinity Principle. In the subatomic realm, for example, physicists can no longer think of an electron as a classical particle following a smooth path in the same way that a planet or a cannonball does. According to quantum mechanics, trajectories become jittery, blurry, and poorly defined at the microscopic scale, so we need to describe the behavior of electrons as probability waves instead of Newtonian trajectories. As soon as we do that, however, calculus returns triumphantly. It governs the evolution of probability waves through something called the Schrödinger equation.

It's incredible but true: Even in the subatomic realm where Newtonian physics breaks down, Newtonian calculus still works. In fact, it works spectacularly well. As we'll see in the pages ahead, it has teamed up with quantum mechanics to predict the remarkable effects that underlie medical imaging, from MRI and CT scans to the more exotic positron emission tomography.

It's time for us to take a closer look at the language of the universe. Naturally, the place to start is at infinity.

1 Infinity

THE BEGINNINGS OF mathematics were grounded in everyday concerns. Shepherds needed to keep track of their flocks. Farmers needed to weigh the grain reaped in the harvest. Tax collectors had to decide how many cows or chickens each peasant owed the king. Out of such practical demands came the invention of numbers. At first they were tallied on fingers and toes. Later they were scratched on animal bones. As their representation evolved from scratches to symbols, numbers facilitated everything from taxation and trade to accounting and census taking. We see evidence of all this in Mesopotamian clay tablets written more than five thousand years ago: row after row of entries recorded with the wedge-shaped symbols called cuneiform.

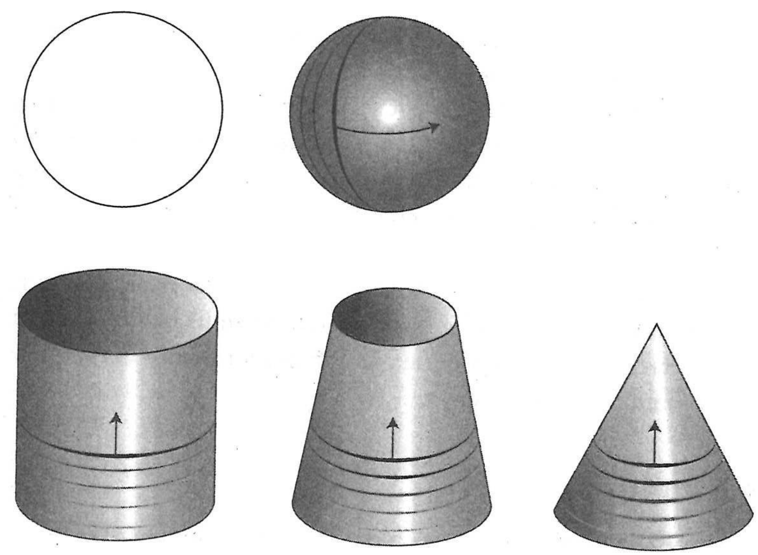
Along with numbers, shapes mattered too. In ancient Egypt, the measurement of lines and angles was of paramount importance. Each year surveyors had to redraw the boundaries of farmers' fields after the summer flooding of the Nile washed the borderlines away. That activity later gave its name to the study of shape in general: *geometry*, from the Greek *gē*, "earth," and *metrēs*, "measurer."

At the start, geometry was hard-edged and sharp-cornered. Its predilection for straight lines, planes, and angles reflected its utilitarian origins—triangles were useful as ramps, pyramids as monuments and tombs, and rectangles as tabletops, altars, and plots of land. Builders and carpenters used right angles for plumb lines. For sailors, architects, and priests, knowledge of straight-line geometry was essential for surveying, navigating, keeping the calendar, predicting eclipses, and erecting temples and shrines.

Yet even when geometry was fixated on straightness, one curve always stood out, the most perfect of all: the circle. We see circles in tree rings, in the ripples on a pond, in the shape of the sun and the moon. Circles surround us in nature. And as we gaze at circles, they gaze back at us, literally. There they are in the eyes of our loved ones, in the circular outlines of their pupils and irises. Circles span the practical and the emotional, as wheels and wedding rings, and they are mystical too. Their eternal return suggests the cycle of the seasons, reincarnation, eternal life, and never-ending love. No wonder circles have commanded attention for as long as humanity has studied shapes.

Mathematically, circles embody change without change. A point moving around the circumference of a circle changes direction without ever changing its distance from a center. It's a minimal form of change, a way to change and curve in the slightest way possible. And, of course, circles are symmetrical. If you rotate a circle about its center, it looks unchanged. That rotational symmetry may be why circles are so ubiquitous. Whenever some aspect of nature doesn't care about direction, circles are bound to appear. Consider what happens when a raindrop hits a puddle: tiny ripples expand outward from the point of impact. Because they spread equally fast in all directions and because they started at a single point, the ripples *have* to be circles. Symmetry demands it.

Circles can also give birth to other curved shapes. If we imagine skewering a circle on its diameter and spinning it around that axis in three-dimensional space, the rotating circle makes a sphere, the shape of a globe or a ball. When a circle is moved vertically into the third dimension along a straight line at right angles to its plane, it makes a cylinder, the shape of a can or a hatbox. If it shrinks at the same time as it's moving vertically, it makes a cone; if it expands as it moves vertically, it makes a truncated cone (the shape of a lampshade).



Circles, spheres, cylinders, and cones fascinated the early geometers, but they found them much harder to analyze than triangles, rectangles, squares, cubes, and other rectilinear shapes made of straight lines and flat planes. They wondered about the areas of curved surfaces and the volumes of curved solids but had no clue how to solve such problems. Roundness defeated them.

Infinity as a Bridge Builder

Calculus began as an outgrowth of geometry. Back around 250 BCE in ancient Greece, it was a hot little mathematical startup devoted to the mystery of curves. The ambitious plan of its devotees was to use infinity to build a bridge between the curved and the straight. The hope was that once that link was established, the methods and techniques of straight-line geometry could be shuttled across the bridge and brought to bear on the mystery of curves. With infinity's help, all the old problems could be solved. At least, that was the pitch.

At the time, that plan must have seemed pretty far-fetched. Infinity had a dubious reputation. It was known for being scary, not useful. Worse yet, it was nebulous and bewildering. What was it exactly? A number? A place? A concept?

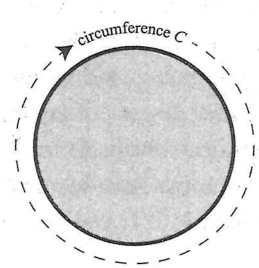
Nevertheless, as we'll see soon and in the chapters to come, infinity turned out to be a godsend. Given all the discoveries and technologies that ultimately flowed from calculus, the idea of using infinity to solve difficult geometry problems has to rank as one of the best ideas anyone ever had.

Of course, none of that could have been foreseen in 250 BCE. Still, infinity did put some impressive notches in its belt right away. One of its first and finest was the solution of a long-standing enigma: how to find the area of a circle.

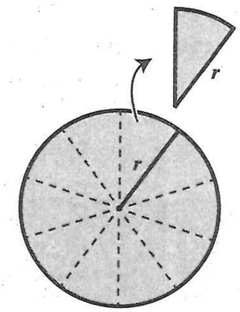
A Pizza Proof

Before I go into the details, let me sketch the argument. The strategy is to reimagine the circle as a pizza. Then we'll slice that pizza into infinitely many pieces and magically rearrange them to make a rectangle. That will give us the answer we're looking for, since moving slices around obviously doesn't change their area from what they were originally, and we know how to find the area of a rectangle: we just multiply its width times its height. The result is a formula for the area of a circle.

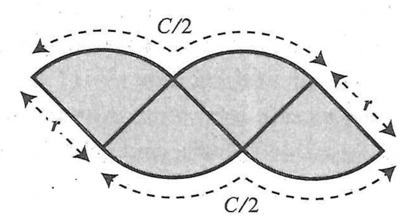
For the sake of this argument, the pizza needs to be an idealized mathematical pizza, perfectly flat and round, with an infinitesimally thin crust. Its circumference, abbreviated by the letter C, is the distance around the pizza, measured by tracing around the crust. Circumference isn't something that pizza lovers ordinarily care about, but if we wanted to, we could measure C with a tape measure.



Another quantity of interest is the pizza's radius, r, defined as the distance from its center to every point on its crust. In particular, r also measures how long the straight side of a slice is, assuming that all the slices are equal and cut from the center out to the crust.



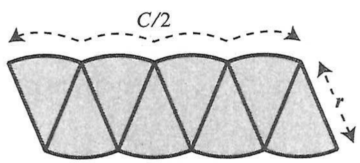
Suppose we start by dividing the pie into four quarters. Here's one way to rearrange them, but it doesn't look too promising.



The new shape looks bulbous and strange with its scalloped top and bottom. It's certainly not a rectangle, so its area is not easy to guess. We seem to be going backward. But as in any drama, the hero needs to get into trouble before triumphing. The dramatic tension is building.

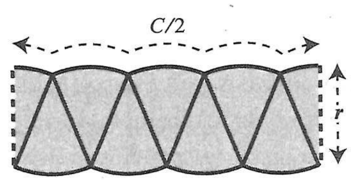
While we're stuck here, though, we should notice two things, because they are going to hold true throughout the proof, and they will ultimately give us the dimensions of the rectangle we're seeking. The first observation is that half of the crust became the curvy top of the new shape, and the other half became the bottom. So the curvy top has a length equal to half the circumference, $C/2$, and so does the bottom, as shown in the diagram. That length is eventually going to turn into the long side of the rectangle, as we'll see. The other thing to notice is that the tilted straight sides of the bulbous shape are just the sides of the original pizza slices, so they still have length r. That length is eventually going to turn into the short side of the rectangle.

The reason we aren't seeing any signs of the desired rectangle yet is that we haven't cut enough slices. If we make eight slices and rearrange them like so, our picture starts to look more nearly rectangular.



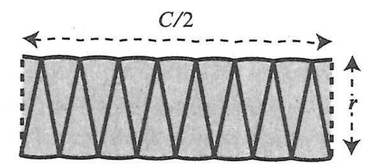
In fact, the pizza starts to look like a parallelogram. Not bad—at least it's almost rectilinear. And the scallops on the top and bottom are a lot less bulbous than they were. They flattened out when we used more slices. As before, they have curvy length $C/2$ on the top and bottom and a slanted-side length r.

To spruce up the picture even more, suppose we cut one of the slanted end pieces in half lengthwise and shift that half to the other side.

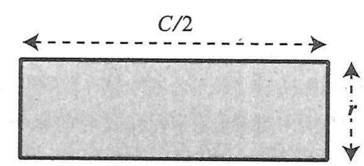


Now the shape looks very much like a rectangle. Admittedly, it's still not perfect because of the scalloped top and bottom caused by the curvature of the crust, but at least we're making progress.

Since making more pieces seems to be helping, let's keep slicing. With sixteen slices and the cosmetic sprucing-up of the end piece, as we did before, we get this result:



The more slices we take, the more we flatten out the scallops produced by the crust. Our maneuvers are producing a sequence of shapes that are magically homing in on a certain rectangle. Because the shapes keep getting closer and closer to that rectangle, we'll call it the *limiting* rectangle.



The point of all this is that we can easily find the area of this limiting rectangle by multiplying its width by its height. All that remains is to find that height and width in terms of the circle's dimensions. Well, since the slices are standing upright, the height is just the radius r of the original circle. And the width is half the circumference of the circle; that's because half of the circumference (the crust of the pizza) went into making the top of the rectangle and the other half got used on the bottom, just as it did at every intermediate stage of working with the bulbous shapes. Thus the width is half the circumference, $C/2$. Putting everything together, the area of the limiting rectangle is given by its height times its width, namely, $A = r \times C/2 = rC/2$. And since moving the pizza slices around did not change their area, this must also be the area of the original circle!

This result for the area of a circle, $A = rC/2$, was first proved (using a similar but much more careful argument) by the ancient Greek mathematician Archimedes (287–212 BCE) in his essay "Measurement of a Circle."

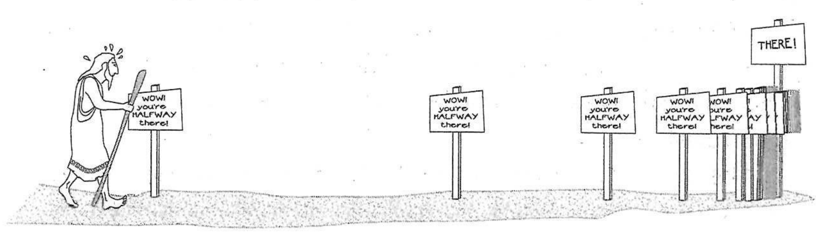
The most innovative aspect of the proof is the way infinity came to the rescue. When we had only four slices, or eight, or sixteen, the best we could do was rearrange the pizza into an imperfect scalloped shape. After an unpromising start, the more slices we took, the more rectangular the shape became. But it was only in the limit of *infinitely* many slices that it became truly rectangular. That's the big idea behind calculus. Everything becomes simpler at infinity.

Limits and the Riddle of the Wall

A limit is like an unattainable goal. You can get closer and closer to it, but you can never get all the way there.

For example, in the pizza proof we were able to make the scalloped shapes more and more nearly rectangular by cutting enough slices and rearranging them. But we could never make them genuinely rectangular. We could only approach that state of perfection. Fortunately, in calculus, the unattainability of the limit usually doesn't matter. We can often solve the problems we're working on by fantasizing that we can actually reach the limit and then seeing what that fantasy implies. In fact, many of the greatest pioneers of the subject did precisely that and made great discoveries by doing so. Logical, no. Imaginative, yes. Successful, very.

A limit is a subtle concept but a central one in calculus. It's elusive because it's not a common idea in daily life. Perhaps the closest analogy is the Riddle of the Wall. If you walk halfway to the wall, and then you walk half the remaining distance, and then you walk half of that, and on and on, will there ever be a step when you finally get to the wall?



The answer is clearly no, because the Riddle of the Wall stipulates that at each step, you walk halfway to the wall, not all the way. After you take ten steps or a million or any other number of steps, there will always be a gap between you and the wall. But equally clearly, you can get arbitrarily close to the wall. What this means is that by taking enough steps, you can get to within a centimeter of it, or a millimeter, or a nanometer, or any other tiny but nonzero distance, but you can never get all the way there. Here, the wall plays the role of the limit. It took about two thousand years for the limit concept to be rigorously defined. Until then, the pioneers of calculus got by just fine with intuition. So don't worry if limits feel hazy for now. We'll get to know them better by watching them in action. From a modern perspective, they matter because they are the bedrock on which all of calculus is built.

If the metaphor of the wall seems too bleak and inhuman (who wants to approach a wall?), try this analogy: Anything that approaches a limit is like a hero engaged in an endless quest. It's not an exercise in total futility, like the hopeless task faced by Sisyphus, who was condemned to roll a boulder up a hill only to see it roll back down again over and over for eternity. Rather, when a mathematical process advances toward a limit (like the scalloped shapes homing in on the limiting rectangle), it's as if a protagonist is striving for something he knows is impossible but for which he still holds out the hope of success, encouraged by the steady progress he's making while trying to reach an unreachable star.

The Parable of .333 . . .

To reinforce the big ideas that everything becomes simpler at infinity and that limits are like unattainable goals, consider the following example from arithmetic. It's the problem of converting a fraction—for example, $\frac{1}{3}$—into an equivalent decimal (in this case, $\frac{1}{3} = 0.333\ldots$). I vividly remember when my eighth-grade math teacher, Ms. Stanton, taught us how to do this. It was memorable because she suddenly started talking about infinity.

Until that moment, I'd never heard a grownup mention infinity. My parents certainly had no use for it. It seemed like a secret that only kids knew about. On the playground, it came up all the time in taunts and one-upmanship.

"You're a jerk!"
"Yeah, well, you're a jerk times two!"
"And you're a jerk times infinity!"
"And you're a jerk times infinity plus one!"
"That's the same as infinity, you idiot!"

Those edifying sessions had convinced me that infinity did not behave like an ordinary number. It didn't get bigger when you added one to it. Even adding infinity to it didn't help. Its invincible properties made it great for finishing arguments in the schoolyard. Whoever deployed it first would win.

But no teacher had ever talked about infinity until Ms. Stanton brought it up that day. Everyone in our class already knew about finite decimals, the familiar kind used for amounts of money, like $10.28, with its two digits after the decimal point. By comparison, infinite decimals, which had infinitely many digits after the decimal point, seemed strange at first but appeared natural as soon as we started to discuss fractions.

We learned that the fraction ⅓ could be written as 0.333 . . . where the dot-dot-dots meant that the threes repeated indefinitely. That made sense to me, because when I tried to calculate ⅓ by doing the long-division algorithm on it, I found myself stuck in an endless loop: three doesn't go into one, so pretend the one is a ten; then three goes into ten three times, which leaves a remainder of one; and now I'm back where I started, still trying to divide three into one. There was no way out of the loop. That's why the threes kept repeating in 0.333 . . . .

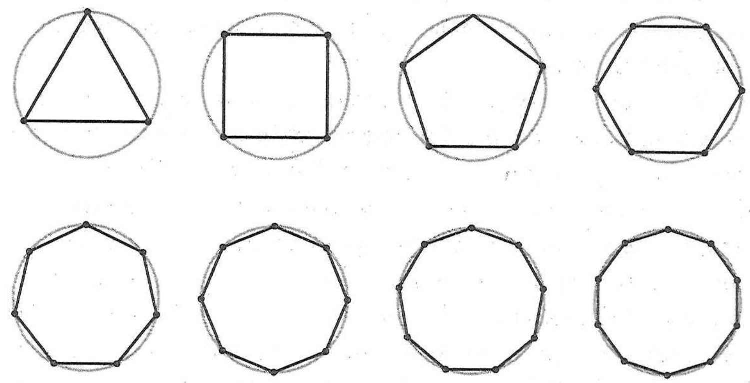
The three dots at the end of 0.333 . . . have two interpretations. The naive interpretation is that there are literally infinitely many 3s packed side by side to the right of the decimal point. We can't write them all down, of course, since there are infinitely many of them, but by writing the three dots we signify that they are all there, at least in our minds. I'll call this the *completed infinity* interpretation. The advantage of this interpretation is that it seems easy and commonsensical, as long as we are willing not to think too hard about what infinity means.

The more sophisticated interpretation is that 0.333 . . . represents a limit, just like the limiting rectangle does for the scalloped shapes in the pizza proof or like the wall does for the hapless walker. Except here, 0.333 . . . represents the limit of the successive decimals we generate by doing long division on the fraction ⅓. As the division process continues for more and more steps, it generates more and more 3s in the decimal expansion of ⅓. By grinding away, we can produce an approximation as close to ⅓ as we like. If we're not happy with $⅓ \approx 0.3$, we can always go a step further to $⅓ \approx 0.33$, and so on. I'll call this the *potential infinity* interpretation. It's "potential" in the sense that the approximations can potentially go on for as long as desired. There's nothing to stop us from continuing for a million or a billion or any other number of steps. The advantage of this interpretation is that we never have to invoke woolly-headed notions like infinity. We can stick to the finite.

For working with equations like ⅓ = 0.333 . . ., it doesn't really matter which view we take. They're equally tenable and yield the same mathematical results in any calculation we care to perform. But there are other situations in mathematics where the completed infinity interpretation can cause logical mayhem. This is what I meant in the introduction when I raised the specter of the golem of infinity. Sometimes it really does make a difference how we think about the results of a process that approaches a limit. Pretending that the process actually terminates and that it somehow reaches the nirvana of infinity can occasionally get us into trouble.

The Parable of the Infinite Polygon

As a chastening example, suppose we put a certain number of dots on a circle, space them evenly, and connect them to one another with straight lines. With three dots, we get an equilateral triangle; with four, a square; with five, a pentagon; and so on, running through a sequence of rectilinear shapes called regular polygons.



Notice that the more dots we use, the rounder the polygons become and the closer they get to the circle. Meanwhile, their sides get shorter and more numerous. As we move progressively further through the sequence, the polygons approach the original circle as a limit.

In this way, infinity is bridging two worlds again. This time it's taking us from the rectilinear to the round, from sharp-cornered polygons to silky-smooth circles, whereas in the pizza proof, infinity brought us from round to rectilinear as it transformed a circle into a rectangle.

Of course, at any finite stage, a polygon is still just a polygon. It's not yet a circle and it never becomes one. It gets closer and closer to being a circle, but it never truly gets there. We are dealing here with potential infinity, not completed infinity. So everything is airtight from the standpoint of logical rigor.

But what if we could go all the way to completed infinity? Would the resulting infinite polygon with infinitesimally short sides actually *be* a circle? It's tempting to think so, because then the polygon would be smooth. All its corners would be sanded off. Everything would become perfect and beautiful.

The Allure and Peril of Infinity

There's a general lesson here: Limits are often simpler than the approximations leading up to them. A circle is simpler and more graceful than any of the thorny polygons that approach it. So too for the pizza proof, where the limiting rectangle was simpler and more elegant than the scalloped shapes, with their unsightly bulges and cusps. And likewise for the fraction $1/3$. It was simpler and more handsome than any of the ungainly fractions creeping up on it, with their big ugly numerators and denominators, like $3/10$ and $33/100$ and $333/1000$. In all these cases, the limiting shape or number was simpler and more symmetrical than its finite approximators.

This is the allure of infinity. Everything becomes better there.

With that lesson in mind, let's return to the parable of the infinite polygon. Should we take the plunge and say that a circle truly *is* a polygon with infinitely many infinitesimal sides? No. We mustn't do that, mustn't yield to that temptation. Doing so would be to commit the sin of completed infinity. It would condemn us to logical hell.

To see why, suppose we entertain the thought, just for a moment, that a circle is indeed an infinite polygon with infinitesimal sides. How long, exactly, are those sides? Zero length? If so, then infinity times zero—the combined length of all those sides—must equal the circumference of the circle. But now imagine a circle of double the circumference. Infinity times zero would also have to equal that larger circumference as well. So infinity times zero would have to be both the circumference and double the circumference. What nonsense! There simply is no consistent way to define infinity times zero, and so there is no sensible way to regard a circle as an infinite polygon.

Nevertheless, there is something so enticing about this intuition. Like the biblical original sin, the original sin of calculus—the temptation to treat a circle as an infinite polygon with infinitesimally short sides—is very hard to resist, and for the same reason. It tempts us with the prospect of forbidden knowledge, with insights unavailable by ordinary means. For thousands of years, geometers struggled to figure out the circumference of a circle. If only a circle could be replaced by a polygon made of many tiny straight sides, the problem would be so much easier.

By listening to the hiss of this serpent—but holding back just enough, by using potential infinity instead of the more tempting completed infinity—mathematicians learned how to solve the circumference problem and other mysteries of curves. In the coming chapters, we'll see how they did it. But first, we need to gain an even deeper appreciation of just how dangerous completed infinity can be. It's a gateway sin to many others, including the sin our teachers warned us about first.

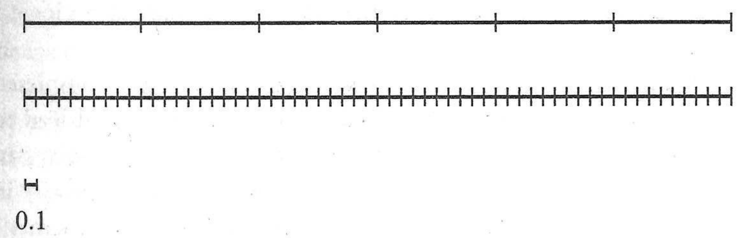
The Sin of Dividing by Zero

All across the world, students are being taught that division by zero is forbidden. They should feel shocked that such a taboo exists. Numbers are supposed to be orderly and well behaved. Math class is a place for logic and reasoning. And yet it's possible to ask simple things of numbers that just don't work or make sense. Dividing by zero is one of them.

The root of the problem is infinity. Dividing by zero summons infinity in much the same way that a Ouija board supposedly summons spirits from another realm. It's risky. Don't go there.

For those who can't resist and want to understand why infinity lurks in the shadows, imagine dividing 6 by a number that's small and getting close to zero, but that isn't quite zero, say something like 0.1. There's nothing taboo about that. The answer to 6 divided by 0.1 is 60, a fairly sizable number. Divide 6 by an even smaller number, say 0.01, and the answer grows bigger; now it's 600. If we dare to divide 6 by a number much closer to zero, say 0.0000001, the answer gets much bigger; instead of 60 or 600, now it's 60,000,000. The trend is clear. The smaller the divisor, the bigger the answer. In the limit as the divisor approaches zero, the answer approaches infinity. That's the real reason why we can't divide by zero. The faint of heart say the answer is undefined, but the truth is it's infinite.

All of this can be visualized as follows. Imagine dividing a 6-centimeter line into pieces that are each 0.1 centimeter long. Those 60 pieces laid end to end make up the original.



Likewise (but I won't attempt to sketch it), that same line can be chopped into 600 pieces that are each 0.01 centimeter or 60,000,000 pieces that are each 0.0000001 centimeter.

If we keep going and take this chopping frenzy to the limit, we are led to the bizarre conclusion that a 6-centimeter line is made up of *infinitely* many pieces of length *zero*. Maybe that sounds plausible. After all, the line is made up of infinitely many points, and each point has zero length.

But what's so philosophically unnerving is that the same argument applies to a line of *any* length. Indeed, there's nothing special about the number 6. We could just as well have claimed that a line of length 3 centimeters, or 49.57, or 2,000,000,000 is made up of infinitely many points of zero length. Evidently, multiplying zero by infinity can give us any and every conceivable result—6 or 3 or 49.57 or 2,000,000,000. That's horrifying, mathematically speaking.

The Sin of Completed Infinity

The transgression that dragged us into this mess was pretending that we could actually *reach* the limit, that we could treat infinity like an attainable number. Back in the fourth century BCE, the Greek philosopher Aristotle warned that sinning with infinity in this way could lead to all sorts of logical trouble. He railed against what he called completed infinity and argued that only potential infinity made sense.

In the context of chopping a line into pieces, potential infinity would mean that the line could be cut into more and more pieces, as many as desired but still always a finite number and all of nonzero length. That's perfectly permissible and leads to no logical difficulties.

What's verboten is to imagine going all the way to a completed infinity of pieces of zero length. That, Aristotle felt, would lead to nonsense—as it does here, in revealing that zero times infinity can give any answer. And so he forbade the use of completed infinity in mathematics and philosophy. His edict was upheld by mathematicians for the next twenty-two hundred years.

Somewhere in the dark recesses of prehistory, somebody realized that numbers never end. And with that thought, infinity was born. It's the numerical counterpart of something deep in our psyches, in our nightmares of bottomless pits, and in our hopes for eternal life. Infinity lies at the heart of so many of our dreams and fears and unanswerable questions: How big is the universe? How long is forever? How powerful is God? In every branch of human thought, from religion and philosophy to science and mathematics, infinity has befuddled the world's finest minds for thousands of years. It has been banished, outlawed, and shunned. It's always been a dangerous idea. During the Inquisition, the renegade monk Giordano Bruno was burned alive at the stake for suggesting that God, in His infinite power, created innumerable worlds.

Zeno's Paradoxes

About two millennia before the execution of Giordano Bruno, another brave philosopher dared to contemplate infinity. Zeno of Elea (c. 490–430 BCE) posed a series of paradoxes about space, time, and motion in which infinity played a starring and perplexing role. These conundrums anticipated ideas at the heart of calculus and are still being debated today. Bertrand Russell called them immeasurably subtle and profound.

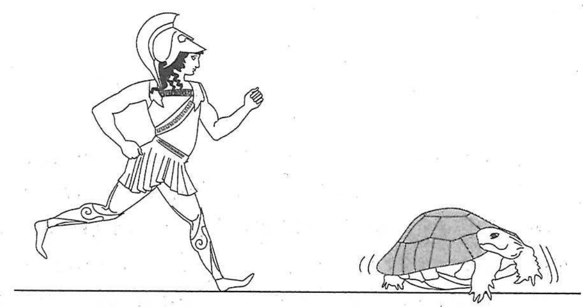
We aren't sure what Zeno was trying to prove with his paradoxes because none of his writings have survived, if any existed to begin

with. His arguments have come down to us through Plato and Aristotle, who summarized them mainly to demolish them. In their telling, Zeno was trying to prove that change is impossible. Our senses tell us otherwise, but our senses deceive us. Change, according to Zeno, is an illusion.

Three of Zeno's paradoxes are particularly famous and strong. The first of them, the Paradox of the Dichotomy, is similar to the Riddle of the Wall but vastly more frustrating. It holds that you can't ever move because before you can take a single step, you need to take a half a step. And before you can do that, you need to take a quarter of a step, and so on. So not only can't you get to the wall—you can't even start walking.

It's a brilliant paradox. Who would have thought that taking a step required completing infinitely many subtasks? Worse still, there is no *first* task to complete. The first task cannot be taking half a step because before that you'd have to complete a quarter of a step, and before that, an eighth of a step, and so on. If you thought you had a lot to do before breakfast, imagine having to finish an infinite number of tasks just to get to the kitchen.

Another paradox, called Achilles and the Tortoise, maintains that a swift runner (Achilles) can never catch up to a slow runner (a tortoise) if the slow runner has been given a head start in a race.



For by the time Achilles reaches the spot where the tortoise started, the tortoise will have moved a little bit farther down the track. And by the time Achilles reaches that new location, the tortoise will have crept slightly farther ahead. Since we all believe that a fast runner *can* overtake a slow runner, either our senses are deceiving us or there is something wrong in the way that we reason about motion, space, and time.

In these first two paradoxes, Zeno seemed to be arguing against space and time being fundamentally continuous, meaning that they can be divided endlessly. His clever rhetorical strategy (some say he invented it) was proof by contradiction, known to lawyers and logicians as *reductio ad absurdum*, reduction to an absurdity. In both paradoxes, Zeno assumed the continuity of space and time and then deduced a contradiction from that assumption; therefore, the assumption of continuity must be false. Calculus is founded on that very assumption and so has a lot at stake in this fight. It rebuts Zeno by showing where his reasoning went wrong.

For example, here's how calculus takes care of Achilles and the tortoise. Suppose the tortoise starts 10 meters ahead of Achilles but Achilles runs 10 times faster, say at a speed of 10 meters per second compared to the tortoise's 1 meter per second. Then it takes Achilles 1 second to make up the tortoise's 10-meter head start. During that time the tortoise will have moved 1 meter farther ahead. It takes Achilles another 0.1 second to make up that difference, by which time the tortoise will have moved another 0.1 meter ahead. Continuing this reasoning, we see that Achilles's consecutive catch-up times add up to the infinite series

$$1 + 0.1 + 0.01 + 0.001 + \cdots = 1.111\ldots \text{ seconds.}$$

Rewritten as an equivalent fraction, this amount of time is equal to $10/9$ seconds. That's how long it takes Achilles to catch up to the tortoise and overtake him. And although Zeno was right that Achilles has infinitely many tasks to complete, there's nothing paradoxical about that. As the math shows, he can do them all in a finite amount of time.

This line of reasoning qualifies as a calculus argument. We just summed an infinite series and calculated a limit, as we did earlier when we discussed why $0.333\ldots = 1/3$. Whenever we work with

infinite decimals, we are doing calculus (even though most people would pooh-pooh it as middle-school arithmetic).

Incidentally, calculus isn't the only way to solve this problem. We could use algebra instead. To do so, we first need to figure out where each runner is on the track at an arbitrary time t seconds after the race begins. Since Achilles runs at a speed of 10 meters per second and since distance equals rate times time, his distance down the track is $10t$. As for the tortoise, he had a head start of 10 meters and he runs with a speed of 1 meter per second, so his distance down the track is $10 + t$. To ascertain the time when Achilles overtakes the tortoise, we have to set those two expressions equal to one another, because that's the algebraic way of asking when Achilles and the tortoise are at the same place at the same time. The resulting equation is

$$10t = 10 + t.$$

To solve this equation, subtract t from both sides. That gives $9t = 10$. Then divide both sides by 9. The result, $t = 10/9$ seconds, is the same as we found with infinite decimals.

So from the perspective of calculus, there really is no paradox about Achilles and the tortoise. If space and time are continuous, everything works out nicely.

Zeno Goes Digital

In a third paradox, the Paradox of the Arrow, Zeno argued against an alternative possibility—that space and time are fundamentally discrete, meaning that they are composed of tiny indivisible units, something like pixels of space and time. The paradox goes like this. If space and time are discrete, an arrow in flight can never move, because at each instant (a pixel of time) the arrow is at some definite place (a specific set of pixels in space). Hence, at any given instant, the arrow is not moving. It is also not moving between instants because, by assumption, there *is* no time between instants. Therefore, at no time is the arrow ever moving.

To my mind, this is the most subtle and interesting of the paradoxes. Philosophers are still debating its status, but it seems to me that Zeno got it two-thirds right. In a world where space and time are discrete, an arrow in flight *would* behave as Zeno said. It would strangely materialize at one place after another as time clicks forward in discrete steps. And he was also right that our senses tell us that the real world is not like that, at least not as we ordinarily perceive it.

But Zeno was wrong that motion would be impossible in such a world. We all know this from our experience of watching movies and videos on our digital devices. Our cell phones and DVRs and computer screens chop everything into discrete pixels, and yet, contrary to Zeno's assertion, motion can take place perfectly well in these discretized landscapes. As long as everything is diced fine enough, we can't tell the difference between a smooth motion and its digital representation. If we were to watch a high-resolution video of an arrow in flight, we'd actually be seeing a pixelated arrow materializing in one discrete frame after another. But because of our perceptual limitations, it would look like a smooth trajectory. Sometimes our senses really do deceive us.

Of course, if the chopping is too blocky, we *can* tell the difference between the continuous and the discrete, and we often find it bothersome. Consider how an old-fashioned analog clock differs from a modern-day digital/mechanical monstrosity. On the analog clock, the second hand sweeps around in a beautifully uniform motion. It depicts time as flowing. Whereas on the digital clock, the second hand jerks forward in discrete steps, *thwack, thwack, thwack.* It depicts time as jumping.

Infinity can build a bridge between these two very different conceptions of time. Imagine a digital clock that advances through trillions of little clicks per second instead of one loud *thwack*. We would no longer be able to tell the difference between that kind of digital clock and a true analog clock. Likewise with movies and videos; as long as the frames flash by fast enough, say at thirty frames a second, they give the impression of seamless flow. And if there were *infinitely* many frames per second, the flow truly would be seamless.

Consider how music is recorded and played back. My younger daughter recently received an old-fashioned Victrola record player for her fifteenth birthday. She's now able to listen to Ella Fitzgerald on vinyl. This is a quintessential analog experience. All of Ella's notes and scats glide just as smoothly as they did when she sang them; her volume goes continuously from soft to loud and everywhere in between, and her pitch climbs just as gracefully from low to high. Whereas when you listen to her on digital, every aspect of her music is minced into tiny, discrete steps and converted into strings of 0s and 1s. Although conceptually the differences are gigantic, our ears can't hear them.

So in everyday life, the gulf between the discrete and the continuous can often be bridged, at least to a good approximation. For many practical purposes, the discrete can stand in for the continuous, as long as we slice things thinly enough. In the ideal world of calculus, we can go one better. Anything that's continuous can be sliced *exactly* (not just approximately) into infinitely many infinitesimal pieces. That's the Infinity Principle. With limits and infinity, the discrete and the continuous become one.

Zeno Meets the Quantum

The Infinity Principle asks us to pretend that everything can be sliced and diced endlessly. We've already seen how useful such concepts can be. Imagining pizzas that can be cut into arbitrarily thin pieces enabled us to find the area of a circle exactly. The question naturally arises: Do such infinitesimally small things exist in the real world?

Quantum mechanics has something to say about that. It's the branch of modern physics that describes how nature behaves at its smallest scales. It's the most accurate physical theory ever devised, and it is legendary for its weirdness. Its terminology, with its zoo of leptons, quarks, and neutrinos, sounds like something out of Lewis Carroll. The behavior it describes is often weird as well. At the atomic scale, things can happen that would never occur in the macroscopic world.

For instance, consider the Riddle of the Wall from a quantum perspective. If the walker were an electron, there's a chance it might walk right through the wall. This effect is known as quantum tunneling. It actually occurs. It's hard to make sense of this in classical terms, but the quantum explanation is that electrons are described by probability waves. Those waves obey an equation formulated in 1925 by the Austrian physicist Erwin Schrödinger. The solution to Schrödinger's equation shows that a small portion of the electron probability wave exists on the far side of an impenetrable barrier. This means there is some small but nonzero probability that the electron will be detected on the far side of the barrier, as if it had tunneled through the wall. With the help of calculus, we can calculate the rate at which such tunneling events occur, and experiments have confirmed the predictions. Tunneling is real. Alpha particles tunnel out of uranium nuclei at the predicted rate to produce the effect known as radioactivity. Tunneling also plays an important role in the nuclear-fusion processes that make the sun shine, so life on Earth depends partially on tunneling. And it has many technological uses; scanning tunneling microscopy, which allows scientists to image and manipulate individual atoms, is based on the concept.

We have no intuition for such events at the atomic scale, being the gargantuan creatures composed of trillions upon trillions of atoms that we are. Fortunately, calculus can take the place of intuition. By applying calculus and quantum mechanics, physicists have opened a theoretical window on the microworld. The fruits of their insights include lasers and transistors, the chips in our computers, and the LEDs in our flat-screen TVs.

Although quantum mechanics is conceptually radical in many respects, in Schrödinger's formulation, it retains the traditional assumption that space and time are continuous. Maxwell made the same assumption in his theory of electricity and magnetism; so did Newton in his theory of gravity and Einstein in his theory of relativity. All of calculus, and hence all of theoretical physics, hinges on this assumption of continuous space and time. That assumption of continuity has been resoundingly successful so far.

But there is reason to believe that at much, much smaller scales of the universe, far below the atomic scale, space and time may ultimately lose their continuous character. We don't know for sure what it's like down there, but we can guess. Space and time might become as neatly pixelated as Zeno imagined in his Paradox of the Arrow, but more likely they'd degenerate into a disorderly mess because of quantum uncertainty. At such small scales, space and time might seethe and roil at random. They might fluctuate like bubbling foam.

Although there is no consensus about how to visualize space and time at these ultimate scales, there is universal agreement about how small those scales are likely to be. They are forced upon us by three fundamental constants of nature. One of them is the gravitational constant, G. It measures the strength of gravity in the universe. It appeared first in Newton's theory of gravity and again in Einstein's general theory of relativity. It is bound to occur in any future theory that supersedes them. The second constant, $\hbar$ (pronounced "h bar"), reflects the strength of quantum effects. It appears, for example, in Heisenberg's uncertainty principle and in Schrödinger's wave equation of quantum mechanics. The third constant is the speed of light, c. It is the speed limit for the universe. No signal of any kind can travel faster than c. This speed must necessarily enter any theory of space and time because it ties the two of them together via the principle that distance equals rate times time, where c plays the role of the rate or speed.

In 1899, the father of quantum theory, a German physicist named Max Planck, realized that there was one and only one way to combine these fundamental constants to produce a scale of length. That unique length, he concluded, was a natural yardstick for the universe. In his honor, it is now called the Planck length. It is given by the algebraic combination

$$\text{Planck length} = \sqrt{\frac{\hbar G}{c^3}}.$$

When we plug in the measured values of G, $\hbar$, and c, the Planck length comes out to be about 10^{-35} meters, a stupendously small distance that's about a hundred million trillion times smaller than the diameter of a proton. The corresponding Planck time is the time it would take light to traverse this distance, which is about 10^{-43}

seconds. Space and time would no longer make sense below these scales. They're the end of the line.

These numbers put a bound on how fine we could ever slice space or time. To get a feel for the level of precision we're talking about here, consider how many digits we would need to make one of the most extreme comparisons imaginable. Take the largest possible distance, the estimated diameter of the known universe, and divide it by the smallest possible distance, the Planck length. That unfathomably extreme ratio of distances is a number with only sixty digits in it. I want to stress that—*only* sixty digits. That's the most we would ever need to express one distance in terms of another. Using more digits than that—say a hundred digits, let alone infinitely many—would be colossal overkill, *way* more than we would ever need to describe any real distances out there in the material world.

And yet in calculus, we use infinitely many digits all the time. As early as middle school, students are asked to think about numbers like 0.333 . . . whose decimal expansion goes on forever. We call these real numbers, but there is nothing real about them. The requirement to specify a real number by an infinite number of digits after the decimal point is exactly what it means to be *not* real, at least as far as we understand reality through physics today.

If real numbers are not real, why do mathematicians love them so much? And why are schoolchildren forced to learn about them? Because calculus needs them. From the beginning, calculus has stubbornly insisted that everything—space and time, matter and energy, all objects that ever have been or will be—should be regarded as continuous. Accordingly, everything can and should be quantified by real numbers. In this idealized, imaginary world, we pretend that everything can be split finer and finer without end. The whole theory of calculus is built on that assumption. Without it, we couldn't compute limits, and without limits, calculus would come to a clanking halt. If all we ever used were decimals with only sixty digits of precision, the number line would be pockmarked and cratered. There would be holes where pi, the square root of two, and any other numbers that need infinitely many digits after the decimal point should exist. Even a simple fraction such as ⅓ would be missing, because it too requires an infinite number of digits (0.333 . . .) to pinpoint its location on the number line. If we want to think of the totality of all numbers as forming a continuous line, those numbers have to be real numbers. They may be an approximation of reality, but they work amazingly well. Reality is too hard to model any other way. With infinite decimals, as with the rest of calculus, infinity makes everything simpler.

Kuta Software - Infinite Algebra 2 Name____________________

Factoring: All Techniques Combined (Hard)

Date________________ Period____

Factor each.

1) $x^3 - 5x^2 - x + 5$

2) $x^4 - 2x^2 - 15$

3) $x^6 - 26x^3 - 27$

4) $x^6 + 2x^4 - 16x^2 - 32$

5) $x^4 - 13x^2 + 40$

6) $x^9 - x^6 - x^3 + 1$

7) $x^6 - 4x^2$

8) $x^4 + 14x^2 + 45$

9) $2x^4 + x^2 - 6$

10) $2x^2 - 13x + 20$

11) $4x^3 - x^2 - 4x + 1$

12) $4x^8 - 61x^4 + 225$

13) $5x^2 + 24x - 5$

14) $5x^2 + 29x + 20$

15) $4x^2 + 4x - 15$

16) $10x^3 - 8x^2 + 25x - 20$

17) $-64x^3 + 125 = 0$

18) $8x^4 + 10x^2 - 3$

Kuta Software - Infinite Algebra 2 Name________________________________

Factoring: All Techniques Combined (Hard)

Date________________ Period____

Factor each.

1) $x^3 - 5x^2 - x + 5$

$(x - 5)(x + 1)(x - 1)$

2) $x^4 - 2x^2 - 15$

$(x^2 - 5)(x^2 + 3)$

3) $x^6 - 26x^3 - 27$

$(x - 3)(x^2 + 3x + 9)(x + 1)(x^2 - x + 1)$

4) $x^6 + 2x^4 - 16x^2 - 32$

$(x^2 + 2)(x^2 + 4)(x + 2)(x - 2)$

5) $x^4 - 13x^2 + 40$

$(x^2 - 5)(x^2 - 8)$

6) $x^9 - x^6 - x^3 + 1$

$(x - 1)^2(x^2 + x + 1)^2(x + 1)(x^2 - x + 1)$

7) $x^6 - 4x^2$

$x^2(x^2 - 2)(x^2 + 2)$

8) $x^4 + 14x^2 + 45$

$(x^2 + 5)(x^2 + 9)$

9) $2x^4 + x^2 - 6$

$(2x^2 - 3)(x^2 + 2)$

10) $2x^2 - 13x + 20$

$(2x - 5)(x - 4)$

11) $4x^3 - x^2 - 4x + 1$

$(4x - 1)(x + 1)(x - 1)$

12) $4x^8 - 61x^4 + 225$

$(2x^2 + 5)(2x^2 - 5)(x^2 + 3)(x^2 - 3)$

13) $5x^2 + 24x - 5$

$(5x - 1)(x + 5)$

14) $5x^2 + 29x + 20$

$(5x + 4)(x + 5)$

15) $4x^2 + 4x - 15$

$(2x - 3)(2x + 5)$

16) $10x^3 - 8x^2 + 25x - 20$

$(5x - 4)(2x^2 + 5)$

17) $-64x^3 + 125 = 0$

$(4x - 5)(-16x^2 - 20x - 25) = 0$

18) $8x^4 + 10x^2 - 3$

$(2x + 1)(2x - 1)(2x^2 + 3)$

Calculus - SUMMER PACKET

NAME:___________________________

Summer + Math = (Best Summer Ever)^2

NO CALCULATOR!!!

Given $f(x) = x^2 - 2x + 5$, find the following.

1. $f(-2) =$

2. $f(x+2) =$

3. $f(x+h) =$

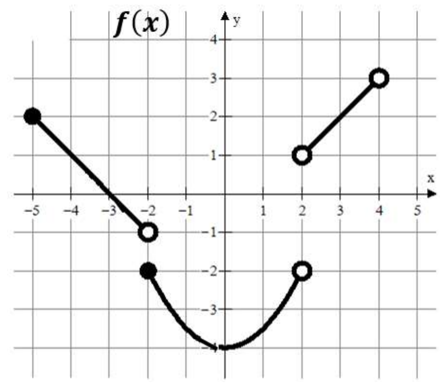
Use the graph $f(x)$ to answer the following.

4. $f(0) =$ $\quad$ $f(4) =$

$f(-1) =$ $\quad$ $f(-2) =$

$f(2) =$ $\quad$ $f(3) =$

$f(x) = 2$ when $x = ?$ $\quad$ $f(x) = -3$ when $x = ?$



Write the equation of the line meets the following conditions. Use point-slope form.
$\boldsymbol{y - y_1 = m(x - x_1)}$

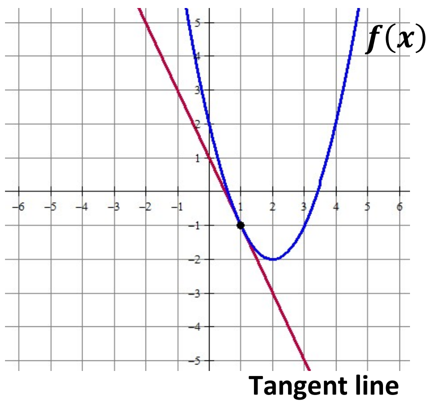
5. slope $= 3$ and $(4, -2)$

6. $m = -\frac{3}{2}$ and $f(-5) = 7$

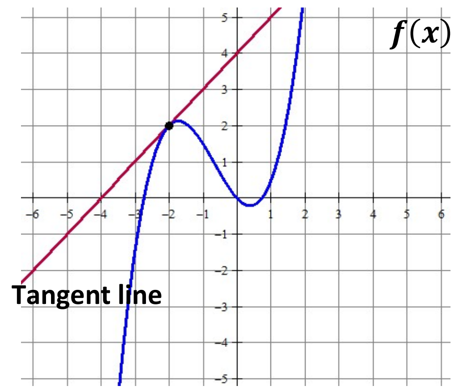
7. $f(4) = -8$ and $f(-3) = 12$

Write the equation of the tangent line in point slope form. $y - y_1 = m(x - x_1)$

8. The line tangent to $f(x)$ at $x = 1$



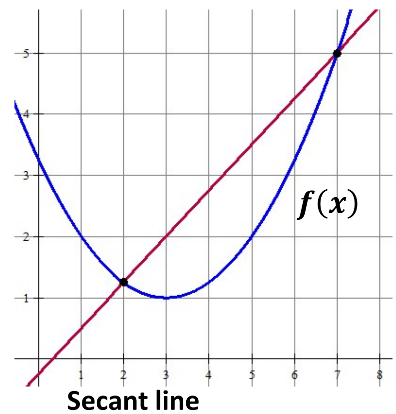
9. The line tangent to $f(x)$ at $x = -2$



MULTIPLE CHOICE! Remember slope = $\frac{y_2 - y_1}{x_2 - x_1}$

10. Which choice represents the slope of the secant line shown?

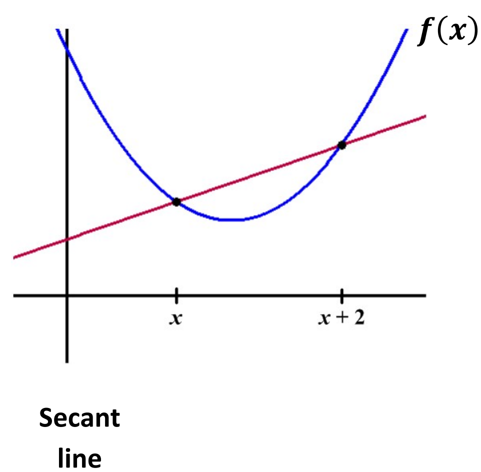
A) $\frac{7-2}{f(7)-f(2)}$ B) $\frac{f(7)-2}{7-f(2)}$ C) $\frac{7-f(2)}{f(7)-2}$ D) $\frac{f(7)-f(2)}{7-2}$



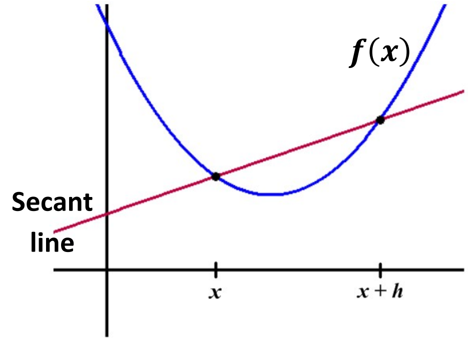
11. Which choice represents the slope of the secant line shown?

A) $\frac{f(x)-f(x+2)}{x+2-x}$ B) $\frac{f(x+2)-f(x)}{x+2-x}$ C) $\frac{f(x+2)-f(x)}{x-(x+2)}$

D) $\frac{x+2-x}{f(x)-f(x+2)}$



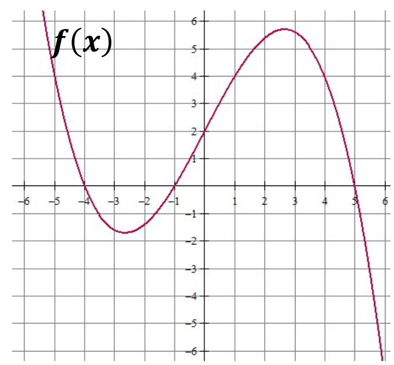
12. Which choice represents the slope of the secant line shown?



A) $\frac{f(x+h)-f(x)}{x-(x+h)}$ B) $\frac{x-(x+h)}{f(x+h)-f(x)}$ C) $\frac{f(x+h)-f(x)}{x+h-x}$

D) $\frac{f(x)-f(x+h)}{x+h-x}$

13. Which of the following statements about the function $f(x)$ is true?

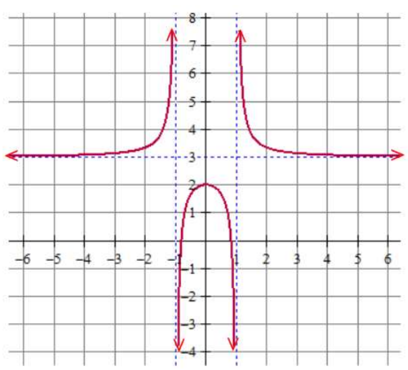


I. $f(2) = 0$
II. $(x + 4)$ is a factor of $f(x)$
III. $f(5) = f(-1)$

(A) I only
(B) II only
(C) III only
(D) I and III only
(E) II and III only

Find the domain and range (express in interval notation). Find all horizontal and vertical asymptotes.

14.



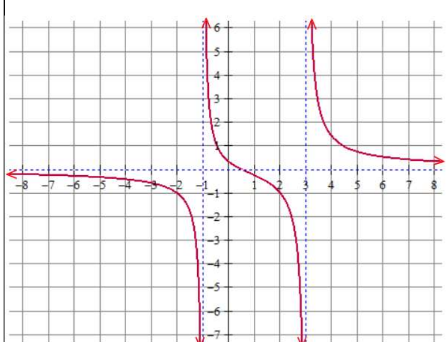
Domain:

Range:

Horizontal Asymptote(s):

Vertical Asymptotes(s):

15.



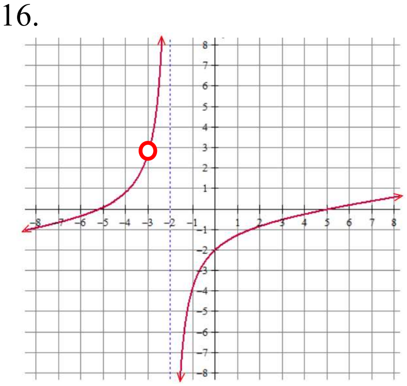
Domain:

Range:

Horizontal Asymptote(s):

Vertical Asymptotes(s):

16.



Domain:

Range:

Horizontal Asymptote(s):

Vertical Asymptotes(s):

MULTIPLE CHOICE!

17. Which of the following functions has a vertical asymptote at $x = 4$?

(A) $\frac{x+5}{x^2-4}$

(B) $\frac{x^2-16}{x-4}$

(C) $\frac{4x}{x+1}$

(D) $\frac{x+6}{x^2-7x+12}$

(E) None of the above

---

18. Consider the function: $(x) = \frac{x^2-5x+6}{x^2-4}$. Which of the following statements is true?

I. $f(x)$ has a vertical asymptote of $x = 2$
II. $f(x)$ has a vertical asymptote of $x = -2$
III. $f(x)$ has a horizontal asymptote of $y = 1$

(A) I only

(B) II only

(C) I and III only

(D) II and III only

(E) I, II and III

Rewrite the following using rational exponents. Example: $\frac{1}{\sqrt[3]{x^2}} = x^{-\frac{2}{3}}$

19. $\sqrt[5]{x^3} + \sqrt[5]{2x}$	20. $\sqrt{x+1}$	21. $\frac{1}{\sqrt{x+1}}$
22. $\frac{1}{\sqrt{x}} - \frac{2}{x}$	23. $\frac{1}{4x^3} + \frac{1}{2}\sqrt[4]{x^3}$	24. $\frac{1}{4\sqrt{x}} - 2\sqrt{x+1}$

Write each expression in radical form and positive exponents. Example: $x^{-\frac{2}{3}} + x^{-2} = \frac{1}{\sqrt[3]{x^2}} + \frac{1}{x^2}$

25. $x^{-\frac{1}{2}} - x^{\frac{3}{2}}$	26. $\frac{1}{2}x^{-\frac{1}{2}} + x^{-1}$	27. $3x^{-\frac{1}{2}}$
28. $(x+4)^{-\frac{1}{2}}$	29. $x^{-2} + x^{\frac{1}{2}}$	30. $2x^{-2} + \frac{3}{2}x^{-1}$

Need to know basic trig functions in RADIANS! We never use degrees. You can either use the Unit Circle or Special Triangles to find the following.

31. $\sin\frac{\pi}{6}$	32. $\cos\frac{\pi}{4}$	33. $\sin 2\pi$
34. $\tan\pi$	35. $\sec\frac{\pi}{2}$	36. $\cos\frac{\pi}{6}$
37. $\sin\frac{\pi}{3}$	38. $\sin\frac{3\pi}{2}$	39. $\tan\frac{\pi}{4}$
40. $\csc\frac{\pi}{2}$	41. $\sin\pi$	42. $\cos\frac{\pi}{3}$
43. Find x where $0 \le x \le 2\pi$, $\sin x = \dfrac{1}{2}$	44. Find x where $0 \le x \le 2\pi$, $\tan x = 0$	45. Find x where $0 \le x \le 2\pi$, $\cos x = -1$

Solve the following equations. Remember $\boldsymbol{e^0 = 1}$ and $\mathbf{\ln 1 = 0}$.

46. $e^x + 1 = 2$	47. $3e^x + 5 = 8$	48. $e^{2x} = 1$
49. $\ln x = 0$	50. $3 - \ln x = 3$	51. $\ln(3x) = 0$
52. $x^2 - 3x = 0$	53. $e^x + xe^x = 0$	54. $e^{2x} - e^x = 0$

Solve the following trig equations where $0 \leq x \leq 2\pi$.		
55. $\sin x = \frac{1}{2}$	56. $\cos x = -1$	57. $\cos x = \frac{\sqrt{3}}{2}$
58. $2\sin x = -1$	59. $\cos x = \frac{\sqrt{2}}{2}$	60. $\cos\left(\frac{x}{2}\right) = \frac{\sqrt{3}}{2}$
61. $\tan x = 0$	62. $\sin(2x) = 1$	63. $\sin\left(\frac{x}{4}\right) = \frac{\sqrt{3}}{2}$

For each function, determine its domain and range.

Function	Domain	Range
64. $y = \sqrt{x-4}$		
65. $y = (x-3)^2$		
66. $y = \ln x$		
67. $y = e^x$		
68. $y = \sqrt{4-x^2}$		

Simplify.

69. $\frac{\sqrt{x}}{x}$	70. $e^{\ln x}$	71. $e^{1+\ln x}$

72. $\ln 1$

73. $\ln e^7$

74. $\log_3 \frac{1}{3}$

75. $\log_{1/2} 8$

76. $\ln \frac{1}{2}$

77. $27^{\frac{2}{3}}$

78. $\left(5a^{2/3}\right)\left(4a^{3/2}\right)$

79. $\frac{4xy^{-2}}{12x^{-\frac{1}{3}}y^{-5}}$

80. $\left(4a^{5/3}\right)^{3/2}$

If $\boldsymbol{f(x) = \{(3,5),(2,4),(1,7)\}}$ $\boldsymbol{g(x) = \sqrt{x-3}}$, $\boldsymbol{h(x) = \{(3,2),(4,3),(1,6)\}}$ $\boldsymbol{k(x) = x^2 + 5}$, then determine each of the following.

81. $(f+h)(1)$

82. $(k-g)(5)$

83. $f\left(h(3)\right)$

84. $g\left(k(7)\right)$

85. $h(3)$

86. $g\left(g(9)\right)$

87. $f^{-1}(4)$

88. $k^{-1}(x)$

89. $k\left(g(x)\right)$

90. $g(f(2))$